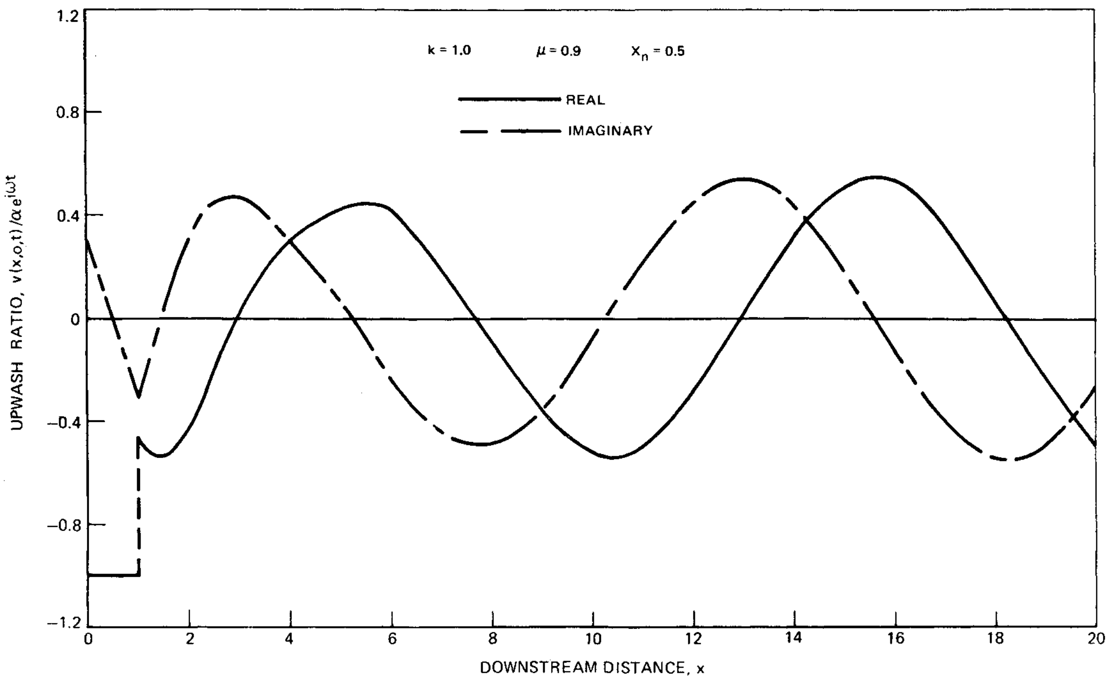

Fig. 2 Upwash on airfoil and wake for a pitching motion.

blade motion. At very low frequencies, the first term on the right-hand side of Eq. (19) dominates, at least in the near wake. This term arises from the counter vorticity shed from the trailing edge of the airfoil and it provides an upwash distribution, $v(x, 0, t)$, having a spatial frequency equal to ω. Additional upwash frequency components arise from the second term in Eq. (19). In particular, the upwash has spatial frequencies approximately equal to $\omega M(M+1)^{-1}$ and $\omega M(M-1)^{-1}$, which correspond to disturbances moving downstream at speeds equal to the freestream speed plus and minus the speed of sound propagation, respectively. The ratio of the wake upwash to airfoil angular displacement is plotted vs distance in Fig. 2 for an airfoil undergoing a pitching motion, $\alpha e^{i\omega t}$, about an axis located at the point $(x_n, 0)$. The predominant spatial frequency of the waves depicted in this figure is the vortex shedding frequency ω.

Once the normal velocity distribution on the airfoil wake is determined, the pressure field downstream of the airfoil can be obtained from Eqs. (3) and (7). Calculations have revealed that the wake upwash exerts an important influence on this pressure field even at points which are of the order of ten chord lengths from the wake.

Conclusions

The closed form solution for the unsteady flowfield produced by an airfoil undergoing harmonic motions in a supersonic stream has been extended to include the region downstream of the airfoil. The essential information required to complete the determination of the unsteady field was the wake pressure distribution. For an oscillating airfoil in a subsonic stream, the wake must be considered to specify uniquely the pressure distribution on the airfoil. This is not the case in supersonic flow. However, if the oscillating airfoil problem is solved in terms of prescribed normal velocities on the airfoil surface and if the upwash in the wake is neglected, a pressure discontinuity will appear at the trailing edge of the airfoil and extend downstream. In addition, large errors will be present in the computed pressure field downstream of the airfoil. These factors are important considerations in aerodynamic interference problems and signal the need for a vortex sheet representing the unsteady wake in potential supersonic flow.

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Higher Order Boundary Layer for Viscous Flow past Sharp Wedges

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Introduction

Two decades have been devoted to finding a second term in the asymptotic expansion of the Navier-Stokes equation for the flat plate case.[1,2,3] They concluded that the second nonzero

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Table 1 Values of $K_i(\xi)$, $\psi_i(\xi,\eta)$, and boundary conditions on Ψ_i

i	K_i	ψ_i	Behavior of Ψ_i as $\sigma \to \infty$ (decay of vorticity)	Boundary conditions on Ψ_i as $\sigma \to \infty$ (from matching)
2	$-\alpha_2 b\xi$	$-\alpha_2 b\xi$	$\partial^2\Psi_2/\partial\sigma^2 = \text{AES}$	$\partial\Psi_2/\partial\sigma = \text{AES}$
3	0	0	$\dfrac{\partial^2\Psi_1}{\partial\xi^2} + b^2\dfrac{\partial^2\Psi_3}{\partial\sigma^2} = \text{AES}$	$\partial\Psi_3/\partial\sigma = \text{AES}$
4	$h(\infty)/b\xi$	$\dfrac{h(\infty)\xi}{b(\xi^2+\eta^2)}$	$\dfrac{\partial^2\Psi_2}{\partial\xi^2} + b^2\dfrac{\partial^2\Psi_4}{\partial\sigma^2} = \text{AES}$	$\partial\Psi_4/\partial\sigma = \text{AES}$
5	0	0	$\dfrac{\partial^2\Psi_3}{\partial\xi^2} + b^2\dfrac{\partial^2\Psi_5}{\partial\sigma^2} = \text{AES}$	$\partial\Psi_5/\partial\sigma = \dfrac{-2h(\infty)\sigma}{(b\xi)^3} + \text{AES}$

term in the expansion must contain a logarithmic term and an undetermined coefficient in order to satisfy the exponential decay of vorticity. Van de Vooren and Dykstra[4] estimated this constant by matching with the leading edge solution. F. Rhyming[5] studied the leading edge wedge flow by linearizing the Navier-Stokes equation in optimal coordinates[6] with the aid of the Faulkner-Skan solution.[7]

This Note treats the wedge flow in Fig. 1, $\beta \neq 0$, in optimal coordinates by employing the method of matched asymptotic expansions.[8] It is shown that the logarithmic term is not required in order to attain an exponentially decaying vorticity. Solutions are found up to the third nonzero term complete with no arbitrary constants.

Governing Equations

In dimensionless optimal coordinates, the Navier-Stokes equation is of the form[5]

$$\left\{\left[\frac{\partial\psi^*}{\partial\eta}\frac{\partial}{\partial\xi} - \frac{\partial\psi^*}{\partial\xi}\frac{\partial}{\partial\eta} + \frac{4\theta}{(\xi^2+\eta^2)}\left(\eta\frac{\partial\psi^*}{\partial\xi} - \xi\frac{\partial\psi^*}{\partial\eta}\right)\right] - \frac{1}{R}\left[\nabla^2 - \frac{8\theta}{(\xi^2+\eta^2)}\left(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}\right) + \frac{(4\theta)^2}{(\xi^2+\eta^2)}\right]\right\}\nabla^2\psi^* = 0$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}, \quad \xi + i\eta = \left(\frac{X}{L} + i\frac{Y}{L}\right)^{1/(2-\beta)}, \quad L = \left(\frac{U}{C}\right)^{1/m},$$

$U = CX^m$, $m = \beta/(2-\beta)$, $R = UL/\nu$, ν = kinematic viscosity

The optimal coordinate system is shown in Fig. 1.

Method of Approach and Solutions

The outer and inner variables are taken to be (ξ,η) and (ξ,σ), respectively, where $\sigma = b\eta R^{1/2}$, $b = (1+2\theta)^{1/2} = (1-\beta)^{1/2}$. The outer and inner expansions, respectively, take the form

$$\psi^*(\xi,\eta;R) = \psi_1 + R^{-1/2}\psi_2 + R^{-1}\psi_3 + R^{-3/2}\psi_4 + \ldots$$

$$\psi^*(\xi,\sigma;R) = R^{-1/2}\Psi_1 + R^{-1}\Psi_2 + R^{-3/2}\Psi_3 + R^{-2}\Psi_4 + \ldots$$

where $\psi_1 = b^2\xi\eta$ is the potential flow solution, $\Psi_1 = b\xi f(\sigma)$ is the classical boundary-layer solution, with f satisfying the Faulkner-Skan equation $f''' + ff'' + \beta(1-f'^2) = 0$ together with the boundary conditions $f(0) = f'(0) = 0$, $f'(\infty) = 1$. Numerical solutions[9] for f are known and large parameter expansion,[10] $f \sim \sigma - \alpha_2 + \text{AES}$§ has been attained. Values of α_2 for all the wedge angles considered are given in Ref. 11.

The outer terms can be shown[11] to be governed by Laplace's equation given by $\nabla^2\psi_i = 0$ together with the boundary conditions

$$\psi_i(\xi,0) = K_i(\xi)$$

ψ_i is finite as $\eta \to \infty$

Solutions for ψ_i and expressions for K_i are given in Table 1. The general equation governing the terms in the inner expansion is given by

$$\frac{\partial^4\Psi_i}{\partial\sigma^4} - \xi f'\frac{\partial^3\Psi_i}{\partial\xi\partial\sigma^2} + f\frac{\partial^3\Psi_i}{\partial\sigma^3} + 4\theta\left(f'\frac{\partial^2\Psi_i}{\partial\sigma^2} + f''\frac{\partial\Psi_i}{\partial\sigma}\right) + \xi f'''\frac{\partial\Psi_i}{\partial\xi} - f''\frac{\partial\Psi_i}{\partial\sigma} = G_i, \quad i = 1,2,3,\ldots \tag{1}$$

where G_i, given in Table 2, are functions of solutions from terms of order $i-1, i-2, \ldots 1$. The inner solutions must satisfy the conditions of vanishing normal and tangential velocity components at the wall. These are expressed as

$$\Psi_i(\xi,0) = 0, \quad \frac{\partial\Psi_i}{\partial\sigma}(\xi,0) = 0$$

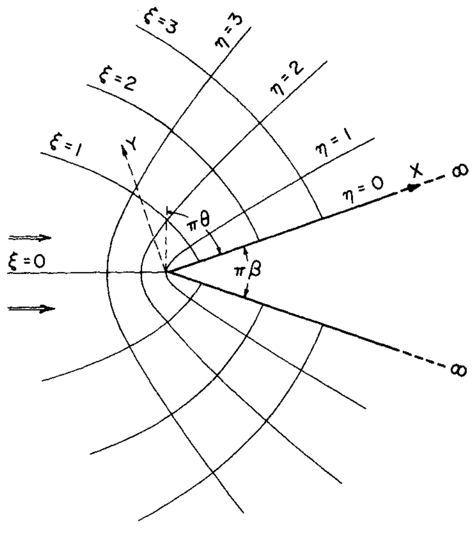

Fig. 1 Optimal coordinate system.

Table 2 G_i, resulting ordinary differential equations

i	ODE for left side of Eq. 1	G_i
2	$g^{iv} + fg''' + 4\theta f'g'' + (4\theta-1)f''g'$	0
3	$\dfrac{1}{b\xi}[h^{iv} + fh''' + (4\theta+1)f'h'' + (4\theta-1)f''h' - f'''h]$	$-\dfrac{1}{b\xi}[\sigma^2 f^{iv} - (8\theta - 16\theta^2)f'' - 8\theta\sigma f''' - \sigma^2 ff'' - 4\theta\sigma ff'']$
4	$\dfrac{1}{(b\xi)^2}[V^{iv} + fV''' + (4\theta+2)f'V'' + (4\theta-1)f''V' - 2f'''V]$	0
5	$\dfrac{1}{(b\xi)^3}[T^{iv} + fT''' + f'T''(3+4\theta) + f''T'(4\theta-1) - 3f'''T - h(\infty)(6f' + 2\sigma f'' + 3\sigma^2 f''' - 8\theta\sigma f'' - 8\theta f')]$	$-\dfrac{1}{(b\xi)^3}[\sigma^2(h^{iv} + f'h'' - h'f'' + fh''' - hf''') - 4\theta\sigma(2h''' + fh'' - hf'') + (6+8\theta)hf' - hh''' + (4\theta+1)h'h'' + 2fh' + (4+16\theta^2+8\theta)h'']$

§ AES stands for Asymptotically Exponentially Small Terms.

Table 3 Results of 3rd- and 5th-order inner solutions

β	$\frac{2}{b^3}h''(0)$	$h'''(0)$	$h(\infty)$	$T''(0)$	$T'''(0)$	$T(\infty)$
0.05	7.831799	−1.129489	24.400880	...	...	...
0.1	3.342832	−0.892131	9.021522	626.100	−37.035363	877.868950
0.2	1.220751	−0.623236	2.538928	30.8200	−8.995353	25.521129
0.3	0.607804	−0.506658	0.971029	6.94458	−3.007645	3.802077
0.4	0.359899	−0.461224	0.421722	2.18406	−1.150627	0.779787
0.5	0.247636	−0.441216	0.197424	0.824847	−0.476491	0.165406
0.6	0.183200	−0.418365	0.090825	0.335752	−0.209822	−0.025134
0.8	0.129017	−0.295316	0.027121	...	...	...
0.9	0.081274	−0.172902	0.119886	...	...	...
0.95	0.046084	−0.093121	0.005844	...	...	...
1.00	0.0	0.0	0.0	...	...	...

In addition, the inner and outer solutions must match at $\eta = 0$ or $\sigma = \infty$ (outer boundary-layer limit) and the exponential decay of vorticity as $\sigma \to \infty$ must be met. These two conditions are given in Table 1.

Equation (1) can be reduced to an ordinary differential equation for solution via the Runge-Kutta Scheme if Ψ_i take special forms. These forms are

$$\Psi_2 = g(\sigma), \quad \Psi_3 = (1/b\xi)[h(\sigma)], \quad \Psi_4 = [1/(b\xi)^2]V(\sigma),$$
$$\Psi_5 = [1/(b\xi)^3][T(\sigma) - h(\infty)\sigma^2]$$

where g, h, V, and T are parameters to be calculated. The resulting ordinary differential equation and boundary conditions are given in Table 2.

In particular, g and V, which are governed by homogeneous equations and boundary conditions, were found to be zero. T, h, T', h' are zero at $\sigma = 0$, T', h' and higher derivatives must vanish exponentially as $\sigma \to \infty$. Values of the initial conditions including $h(\infty)$ and $T(\infty)$ are given in Table 3.

The skin friction and pressure made dimensionless by $\frac{1}{2}\rho U(X)^2$ can be shown to be given by, respectively,

$$C_f = 2f''(0)/bR_X{}^{1/2} + 2h''(0)/b^3R_X{}^{3/2} + 2[T''(0) - 2h(\infty)]/b^5R_X{}^{5/2}$$

and

$$C_p = -h'''(0)/4\theta b^2 R_X - T'''(0)/(2+4\theta)b^4 R_X{}^2$$

where $R_X = [U(X)X/\nu]$ is based on local velocity at X. The dimensionless pressure does not include the inviscid pressure.

Conclusion

The results obtained here are not valid around the leading edge. The numerical solution of the Navier-Stokes equation which includes the leading edge has been found.[11] The skin-friction coefficient curves shown in Fig. 2 have been terminated as $R_X \to 0$. These curves approach the classical boundary-layer values as $R_X \to \infty$.

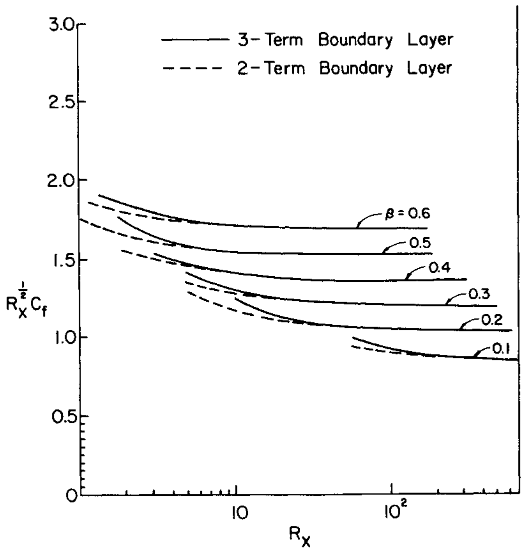

Fig. 2 Variations of $R_x{}^{1/2}C_f$ with R_x for the asymptotic solutions.

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Contribution of a Wall Shear Stress to the Magnus Effect on Nose Shapes

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IN recent years, several authors have calculated the Magnus effect on bodies of differing shapes[1–3] with either fully laminar or fully turbulent boundary layers. These analyses suggest that

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